
On the embedding Menger n-groupoids in n-ary topological groups

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Abstract: In [1] Regilhof proved that the binary cancellative ,abelian, locally compact semigroup when all of whose intern translations are open and continuous, is topologically embeddable in a locally compact binary topological group as an open subsemigroup. In this paper we prove a similar result for Menger n-groupoid $\langle X,(\ )\rangle$ with the topology τ_X . And we concluded the continuity of the n-ary operation ( ) in all arguments in topology τ_X .

Introduction

Topological n-groups, n-semigroups were investigated by many authors. For example, Mukhin and Dudek in [2] proved that the cancellative semiabelian n-semigroup can be embedded in a topological group. In [4] is described a method of embedding topological abelian n-semigroup in topological n-group.

In this paper, in the first time, we find the necessary and sufficient conditions for the topologically embedding a Menger n-groupoid with topology into the n-ary topological group.

Preliminaries

If X is a nonempty set the mapping

$$(\ ):\ X^n \to X$$
$$(x_1,\ldots,x_n)\mapsto (x_1^n)$$

is called an n-ary operation, where X^n is the n-th Cartesian power of X . The number n is called a rank of the operation. In many papers n-ary operation are called n-place function. A nonempty set X together with one n-ary operation ( ) is called an n-ary groupoid (briefly: n-groupoid) and is denoted by $\langle X,(\ )\rangle$.

The n-ary operation ( ) is called associative if

$$((x_1^n)x_{n+1}^{2n-1}) = (x_1^j(x_{j+1}^{j+n})x_{j+n+1}^{2n-1}),\ j=1,2,\ldots,n-1$$

holds for all sequence $x_1^{2n-1}\in X^{2n-1}$.

The n-ary groupoid $\langle X,(\ )\rangle$, where the n-ary operation ( ) is associative, is called an n-ary semigroup or n-semigroup. And if for all consequence $a_1^{n-1}b\in X^n$ the equations $(xa_1^{n-1})=b$ and $(a_1^{n-1}y)=b$ has a solutions on X , the n-semigroup $\langle X,(\ )\rangle$ is called an n-ary group (briefly: n-group) or a polyadic group.

The n-groupoid $\langle X,(\ )\rangle$ is called a Menger n-groupoid if the identity

$$(*)\ ((x_1^n)y_1^{n-1}) = (x_1(x_2y_1^{n-1})\ldots(x_ny_1^{n-1}))\ \forall x_i,y_j\in X\ ,$$
$$i=\overline{1,n},\ j=\overline{1,n-1}$$

is realized.

We say that the n-groupoid $\langle X,(\ )\rangle$ is abelian if

$(x_1 \cdot x_2 \cdot \ldots \cdot x_n) = (x_{\sigma(1)} \cdot x_{\sigma(2)} \cdot \ldots \cdot x_{\sigma(n)})$ for all sequence $x_1^n \in X^n$ and all permutation σ on $\{1,2,\ldots,n\}$.

If the n-ary operation ( ) is continuous, in all variables together, in the topology τ defined on a n-semigroup $\langle X,(\ )\rangle$ then the triple $\langle X,(\ ),\tau\rangle$ is called a topological n-semigroup or n-ary topological semigroup.

The triple $\langle X,(\ ),\tau\rangle$ is called a topological n-group or n-ary topological group if $\langle X,(\ )\rangle$ is an n-group, the operation ( ) is continuous and the solution of each equations $(xa_1^{n-1}) = b$ and $(a_1^{n-1}x) = b$ is continuously depend on $b_1^{n-1}b \in X^n$. This definition is equivalent the definition topological n-group which gives Rusakov S.A.(cf.[3]).

The triple $\langle X,(\ ),\tau\rangle$ is called a topological Menger n-groupoid if the following operation

$$\alpha: \quad X^{2n-1} \to X$$
$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}) \mapsto ((x_1^n) y_1^{n-1})$$

Is continuous, in all variables together, in the topology τ.

$\langle X,(\ ),\tau_X\rangle$ - an Menger n-groupoid $\langle X,(\ )\rangle$ with a topology τ_X is said to be topologically imbedded in a topological m-group $\langle Y,[\ ],\tau_Y\rangle$, n=k(m-1)+1, if there exists an injective mapping $p: X \to Y$ such that $\forall x_1^n \in X^n,\ p((x_1 \cdot x_2 \cdot \ldots \cdot x_n)) = [p(x_1) \cdot p(x_2) \cdot \ldots \cdot p(x_n)]$ (i.e. p is an algebraically imbedded) and p is bicontinuous from X to $p(X)$ subspace to topological space $\langle Y,\tau_Y\rangle$.

In [6] notes that in a Menger n-groupoid $\langle X,(\ )\rangle$ the operation $x \cdot y = \left(x \overset{n-1}{y}\right)$ is associative. The semigroup $\langle X,\cdot\rangle$ obtained in this way is called a diagonal semigroup of $\langle X,(\ )\rangle$.

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Results

Theorem 1. Let $\langle X,(\ ),\tau_X\rangle$ be a Menger n-groupoid with a topology τ_X such that each right translation $x \mapsto \left(x \overset{n-1}{a}\right)$, $\forall a \in X$, is injective and open. The operation α is continuous in $\langle X,\tau_X\rangle$ if and only if the n-ary operation ( ) is continuous in $\langle X,\tau_X\rangle$.

Proof. Let $a_1^n \in X^n$, and let $W((a_1^n))$ be an open neighborhood of the point (a_1^n). Then $\left[W((a_1^n))\overset{n-1}{a}\right]$ is a neighborhood of the point $\left((a_1^n)\overset{n-1}{a}\right)$, where $a \in X$. If the operation α is continuous, then there exists open neighborhoods $U_i(a_i)$ of the point a_i, $i \in \{1,2,\ldots,n\}$, and open neighborhood $U(a)$ of the point a, where $((x_1^n)y_1^{n-1}) \in \left[W((a_1^n))\overset{n-1}{a}\right]$ such that $x_i \in U_i(a_i), i \in \{1,2,\ldots,n\}$ and $y_j \in U(a)$, $j \in \{1,2,\ldots,n-1\}$. In particular we have $\left((x_1^n)\overset{n-1}{a}\right) \in \left[W((a_1^n))\overset{n-1}{a}\right]$. As the translation $x \mapsto \left(x \overset{n-1}{a}\right)$ is injective, then $(x_1^n) \in W((a_1^n))$. Then the n-ary operation ( ) is continuous. The other direction is clear.

Theorem 2. Let $\langle X,(\ ),\tau_X\rangle$ be an associative Menger n-groupoid derived from its commutative diagonal semigroup and let τ_X be a Hausdorff locally compact topology on X. If each translation $x \mapsto \left(a_1^{i-1} x a_{i+1}^n\right)$, $\forall a_i \in X$, $i=\overline{1,n-1}$, is continuous ,open and injective then there exists a locally compact binary topological group $\langle B,+,\tau_B\rangle$ and a topologically imbedded $p: X\to B$ such that $p(X)$ is an open subset of B.

Proof. Let $\langle X,(\ ),\tau_X\rangle$ be a an associative Menger n-groupoid derived from its commutative diagonal semigroup $\langle X,\cdot\ \rangle$. Then by [6] we have $\left(x_1^n\right) = x_1\cdot x_2\cdot\ldots\cdot x_n$, for all sequence $a_1^n \in X^n$. And by [2] there exists a commutative group $\langle B,+\rangle$ and an injection $p:X\to B$ such that $p((a_1\cdot a_2\cdot\ldots\cdot a_n)) = p(a_1)+p(a_2)+\ldots+p(a_n)$, $a_1^n\in X^n$. For all $x\in X$ let $F(x)$ a family of neighborhood of a point x be a filter on X . We choice an arbitrary point $x_0\in X$ and for all $x\in B$ let xF be a filter on B generated by a basis for the filter $\beta = \{p(U)-p(x_0)+x \ / \ U\in F(x_0)\}$.

In the first we proves that if $x\in X$, then $p(x)F$ is generated by a basis for the filter $\{p(U) \ / \ U\in F(x_0)\}$. Indeed, if $U\in F(x_0)$, then $\left(U \overset{n-1}{x}\right) \in F\left(x_0 \overset{n-1}{x}\right)$ because all of translations of X are open, such that $\left(U\overset{n-1}{x}\right) = \left\{\left(u\overset{n-1}{x}\right) \ / \ u\in U\right\}$. The translation $t\mapsto \left(t x_0 \overset{n-2}{x}\right)$ is continuous on the point $x\in X$. Hence, there exists an open neighborhood $V\in F(x)$ such that $\left(V x_0 \overset{n-2}{x}\right) \subset \left(U\overset{n-1}{x}\right)$. Therefore $p(V)+p(x_0)+(n-2)p(x) \subset p(U)+(n-1)p(x)$. Consequently we have $p(V)\subset p(U)-p(x_0)+p(x) \in p(x)F$. Conversely, if $V\in F(x)$, then $\left(V\overset{n-1}{x_0}\right) \in F\left(x\overset{n-1}{x_0}\right)$ and again using the continuity of the translation $t\mapsto \left(t x_0 \overset{n-2}{x}\right)$ on the point x_0, we will show that there exists $U\in F(x_0)$ such that $\left(Ux\overset{n-2}{x_0}\right) \subset \left(V\overset{n-1}{x_0}\right)$. Therefore, $p(U)+p(x)+(n-2)p(x_0)\subset p(V)+(n-1)p(x_0)$ or $p(U)-p(x_0)+p(x)\subset p(V)$. Consequently, $p(x)F$ is generated by a basis for the filter $\{p(U) \ / \ U\in F(x_0)\}$. Is clear that $x\in V$ for all V belongs to xF .

Now for all V belongs to xF we choice an open neighborhood U of the point x_0 on X such that $W = p(U)-p(x_0)+x\subset V$. If $y\in W$,then for certain $u\in U$, we have $y=p(u)-p(x_0)+x$. Let $V'\in F(x_0)$ such that $\left(V' u \overset{n-2}{x_0}\right) \subset \left(U\overset{n-1}{x_0}\right)$. Then $p(V')+p(u)+(n-2)p(x_0)\subset p(U)+(n-1)p(x_0)$. Therefore , we have $p(V')-p(x_0)+y = p(V')-p(x_0)+p(U)-p(x_0)+x\subset p(U)-p(x_0)+x=W$. Consequently, $W\in yF$. As $V\supset W$, then $V\in yF$ for all $y\in W$. By [4,proposition1, §1,chap.1] on the set B there exists a unique topology τ_B ,such that xF a family of neighborhood of a point x be a filter on B.
By this construction we conclude that the mapping $x\mapsto x+y$ is continuous on B for all y belongs to B.
Now we proves that $\langle B,\tau_B\rangle$ is Hausdorff space. Let $y\in B$, $y\neq e$, e is the unit element of the group $\langle B,+\rangle$. If

$y \neq p(x) - p(z)$, $\forall\ x, z \in X$,then $p(U) - p(x_0) \cap p(V) - p(x_0) + y = \phi$, where U, V are the arbitrary neighborhood from $F(x_0)$. Therefore , we have disjoint neighborhoods of points e and y. If $y = p(x) - p(z)$, $\forall\ x, z \in X$,then $x \neq z$ and hence $\overset{n-1}{x_0} x \neq \overset{n-1}{x_0} z$ .As $\langle X, \tau_X \rangle$ a Hausdorff topological space ,then there exists U and V belongs to $F(x_0)$ such that $\left(V \overset{n-2}{x_0} x \right) \cap \left(U \overset{n-2}{x_0} z \right) = \phi$. Hence $p(V) + (n-2)p(x_0) + p(x) \cap p(U) + (n-2)p(x_0) + p(z) = \phi$. Hence $p(V) - p(x_0) + y \cap p(U) - p(x_0) = \phi$. Then , again in this case , the points e and y are the disjoint neighborhoods. By the continuity of the translations on B we conclude that topology τ_B is Hausdorff. So $\langle B, \tau_B \rangle$ is Hausdorff space.

Now let τ be a locally compact topology. As $p(x)F$ is generated by a basis for the filter $\{p(U)\ /\ U \in F(x_0)\}$ for all $x \in X$, then $p(X)$ is an open subset of B and the topology τ_B, induced on $p(X)$, image of the topology τ_X by the mapping p .Therefore, p is a homeomorphism between X and $p(X)$. Hence , each point of the space $\langle X, \tau_X \rangle$ has a compact neighborhood. Therefore, τ_B is a locally compact topology.
Finally, by Ellis theorem [5,th.2] we conclude that $\langle B, +, \tau_B \rangle$ is topological group. Ended proof theorem.
From the previous theorems we obtain the next corollary.

Corollary 1 . Let $\langle X, (\ ), \tau_X \rangle$ be an abelian n-semigroup and let τ_X be a Hausdorff locally compact topology on X . If each translation $x \mapsto \left(a_1^{i-1} x a_{i+1}^n\right)$, $\forall a_i \in X$, $i = \overline{1, n-1}$, of $\langle X, (\ ), \tau_X \rangle$, is continuous ,open and injective then there exists a locally compact binary topological group $\langle B, +, \tau_B \rangle$ and a topologically imbedded $p: X \to B$ such that $p(X)$ is an open subset of B.

Theorem 3 . Let $\langle X, (\ ), \tau_X \rangle$ be an associative Menger n-groupoid derived from its commutative diagonal semigroup with a Hausdorff locally compact topology τ_X such that each translation of $\langle X, (\ ), \tau_X \rangle$ is continuous ,open and injective. Then there exists a locally compact topological abelian group $\langle G, [\ ], \tau_G \rangle$ and a topologically imbedded $p: X \to G$ such that $p(X)$ is an open subset of G.

Proof. Such as in theorem 2 above , the triple $\langle G, [\ ], \tau_G \rangle$, is necessary to keep $G = B$, for all sequence $x_1^n \in G^n$, $[x_1 \cdot x_2 \cdot \ldots \cdot x_n] = x_1 + x_2 + \ldots + x_n$ and $\tau_G = \tau_B$ and p is the same p in the theorem 2.

Corollary 2. Let $\langle X, (\ ), \tau_X \rangle$ be a Menger n-groupoid with a topology τ_X, whish satisfy conditions theorem 2. Then the operation α and the n-ary operation ( ) are continuous on $\langle X, \tau_X \rangle$.

Corollary 3. Let $\langle X, (\ ), \tau_X \rangle$ be a Menger n-groupoid with a topology τ_X, whish satisfy conditions corollary 1. Then the n-ary operation ( ) is continuous **.**

We say that the Menger n-groupoid $\langle X, (\ ) \rangle$ is i-solvable , if the equation $\left(x a_1^{n-1}\right) = b$ and the equation $\left(a_1^i x a_{i+1}^{n-1}\right) = b$ has the unique solution in $\langle X, (\ ) \rangle$, $\forall b \in X$, $\forall a_i \in X$, $i = \overline{1, n-1}$.

Theorem 4 . Let $\langle X, (\ ), \tau_X \rangle$ be an associative Menger n-groupoid i-solvable with a Hausdorff locally compact topology τ_X such that each translation of $\langle X, (\ ), \tau_X \rangle$ is continuous ,open and injective. Then there exists a locally compact topological abelian group $\langle G, [\ ], \tau_G \rangle$ and a topologically imbedded $p: X \to G$ such that $p(X)$ is an open subset of G.

Proof. Let $\langle X, (\ ), \tau_X \rangle$ be an associative Menger n-groupoid i-solvable. Then by [6,cor.3.3.] the Menger n-groupoid $\langle X, (\ ) \rangle$ derived from its commutative diagonal semigroup. And by a corollary 1 above we get to the desired result.

Remark 1 .In general the continuity of the operation α does not lead to the continuity of the n-ary operation ( ). The next example illustrates this.

Let $X =]1, +\infty[$, $n = 3$ and the 3-ary operation ( ) be an ordinary sum of 3 numbers [i.e. $(x_1^3) = x_1 + x_2 + x_3$]. The topology τ on X is the collection of all sets as a union of elements of the ordinary topology on $]1, +\infty[\setminus [2, 5]$ and of elements of the discrete topology on $[2, 5]$. We show that the operation α is continuous , while the ternary operation ( ) is discontinuous , where $\alpha(x_1, x_2, x_3, y_1, y_2) = ((x_1^3) y_1^2)$, $\forall x_1, x_2, x_3, y_1, y_2 \in X$.

Let $x_1, x_2, x_3, y_1, y_2 \in X$. Then $x = x_1 + x_2 + x_3 + y_1 + y_2 > 5$. Let V be an open neighborhood of a point x. Hence $\exists \varepsilon > 0$ such that $]x - \varepsilon, x + \varepsilon[\subset V$.

Intervals $U_i = X \cap \left] x_i - \frac{\varepsilon}{5}, x_i + \frac{\varepsilon}{5} \right[$, $i = 1,2,3$,

$U_4 = X \cap \left] y_1 - \frac{\varepsilon}{5}, y_1 + \frac{\varepsilon}{5} \right[$,

$U_5 = X \cap \left] y_2 - \frac{\varepsilon}{5}, y_2 + \frac{\varepsilon}{5} \right[$ are neighborhoods corresponding points x_1, x_2, x_3, y_1, y_2. It is clear that $U_1 + U_2 + U_3 + U_4 + U_5 \subset V$. Hence α is continuous.

Show that the ternary operation ( ) is discontinuous in the point $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$.

The set $W = \{4\}$ is a neighborhood of the point $4 = \frac{4}{3} + \frac{4}{3} + \frac{4}{3}$. Each neighborhood U_1, U_2 , U_3 of the point $\frac{4}{3}$ completely contains the interval containing the point $\frac{4}{3}$, which is why $U_1 + U_2 + U_3$ is not included in W. Hence the ternary operation ( ) is discontinuous in the point considered.

Remark 2 .If $\langle X, \cdot \rangle$ is a diagonal semigroup of a Menger n-groupoid $\langle X, (\ ) \rangle$,and the n-ary operation ( ) is continuous on $\langle X, \tau \rangle$, then is evident that the binary operation $<<\cdot>>$ while be continuous. We show that in the example given above the binary operation $<<\cdot>>$ of diagonal semigroup is not continuous, while the operation α is continuous.

Indeed, let $x = y = \frac{4}{3}$, the $W = \{4\}$ set is a neighborhood of the point $x \cdot y = \left(x \overset{2}{y} \right) = 4$,but for all neighborhoods $U(x)$ and $U(y)$ of corresponding points x and y. We have $U(x) \cdot U(y) \not\subset W$, because each neighborhood $U(x)$ and $U(y)$ has a completely open interval containing the point $\frac{4}{3}$. Hence the binary operation $<<\cdot>>$ is discontinuous in the point $\left(\frac{4}{3}, \frac{4}{3}\right)$.

Through the above theorem 1 easily we get the following theorem.

Theorem 5. Let $\langle X, (\ ), \tau_X \rangle$ be a Menger n-groupoid with a topology τ_X, where the operation α is continuous. If each right translation $x \mapsto \left(x \overset{n-1}{a} \right)$, $\forall a \in X$, is injective and open, then the diagonal semigroup $\langle X, \cdot, \tau_X \rangle$ is a topological semigroup.

حول إدماج n - زميرة مينكر في الـ n- زمر الطوبولوجية

حمزة ابوجوف

قسم الرياضيات، كلية العلوم ،جامعة سبها

الملخص

برهن ريجيلهوف (Regilhof) أن نصف الزمرة الثنائية القابلة للإختزال ، الآبلية ، المتراصة موضعيا ، حيث كل الإزاحات الداخلية مفتوحة ومستمرة ، يمكن إدماجها طوبولوجيا في زمرة ثنائية طوبولوجية متراصة موضعيا كنصف زمرة جزئية مفتوحة. بهذه الورقة البحثية برهنا نتيجة مماثلة في حالة n- زميرة مينكر $\langle X, ( ) \rangle$ معرف عليها طوبولوجي τ_X . و استنتجنا استمرارية الـ n –عملية ( ) في الطوبولوجي τ_X بالنسبة لكل السعات.

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