New subclass of harmonic univalent functions with respect to 2k-symmetric conjugate points

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Abstract: Let S_H denote the class of functions $f = h + \bar{g}$ which are harmonic univalent and sense preserving in the unit disk is $U = \{z : |z| < 1\}$. We introduce here new classes of harmonic functions with respect to 2k-symmetric conjugate points. Various properties are obtained.

Keywords: Univalent functions, 2k-symmetric conjugate points. Hadamard product (or convolution).

Introduction

A continuous functions $f = u + iv$ is a complex harmonic function in a complex domain

$\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subseteq \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D. See [1].

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in S_H$, **we may express the analytic** functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1 \qquad (1.1)$$

Observe that S_H reduces to S, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

A function f is said to be harmonic starlike of order α in U denoted by $S_H^*(\alpha)$

(see [6]) if $\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right)=\Re\left\{\frac{\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right\}=\Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right\}\geq\alpha$,

For $0\leq\theta\leq 2\pi,\ 0\leq r\leq 1$ and $z\in U$.

Also, a function f is said to be harmonic convex of order α in U denoted by $C_H^*(\alpha)$ (see[6]) if $\frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right)\right\}=\Re\left\{\frac{\partial}{\partial\theta}\log\left(\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)\right)\right\}$

$=\Re\left\{\frac{zh''(z)+h'(z)-\overline{zg''(z)+g'(z)}}{h'(z)+\overline{g'(z)}}\right\}\geq\alpha$ For $0\leq\theta\leq 2\pi,\ 0\leq r\leq 1$ and $z\in U$.

The authors in [1] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on S_H and its subclasses such that [5], [7] and [2] studied the harmonic univalent functions.

Luo and Wang [3] once introduced subclasses of functions in A with respect to 2k-symmetric conjugate points, which satisfy the following inequality:

$$\Re\left\{\frac{\frac{zf'(z)}{f_{2k}(z)}}{\lambda\frac{zf'(z)}{f_{2k}(z)}+(1-\lambda)}\right\}>\alpha\qquad(z\in U),\qquad(1.2)$$

Where $0\leq\alpha<1,\ 0\leq\lambda<1$ and $k\geq 2$ is a fixed positive integer.

Extending the definition (1.2) to include the harmonic functions, we let $\mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$ denote the class of complex-valued, sense-preserving harmonic functions f of the form (1.1), which satisfy the condition

$$\Re\left\{\frac{\frac{\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f_{2k}\left(re^{i\theta}\right)}}{\lambda\frac{\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f_{2k}\left(re^{i\theta}\right)}+(1-\lambda)}\right\}>\alpha\qquad(z\in U),\qquad(1.3)$$

where $k\geq 2$ is a fixed positive integer and $z=re^{i\theta},\ 0\leq\theta\leq 2\pi,\ 0\leq r<1,\ 0\leq\alpha<1$ and $f_{2k}=h_{2k}+\overline{g_{2k}}$ where

$$h_{2k}=\frac{1}{2k}\sum_{v=0}^{k-1}\left[\varepsilon^{-v}h\left(\varepsilon^{-v}z\right)+\varepsilon^{-v}\overline{h\left(\varepsilon^{-v}\overline{z}\right)}\right],g_{2k}=\frac{1}{2k}\sum_{v=0}^{k-1}\left[\varepsilon^{-v}g\left(\varepsilon^{-v}z\right)+\varepsilon^{-v}\overline{g\left(\varepsilon^{-v}\overline{z}\right)}\right],$$

$$\left(\varepsilon=\exp\left(2\pi i/k\right)\right). \qquad (1.4)$$

Also, note that $Q_{H_{sc}}^{(k)}(\lambda,\alpha)$ denote the class of complex-valued, sense-preserving harmonic univalent functions f of the form (1.1), which satisfy the condition:

$$\Re\left\{\frac{\dfrac{\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)\right)}{\frac{\partial}{\partial\theta}f_{2k}\left(re^{i\theta}\right)}}{\lambda\dfrac{\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)\right)}{\frac{\partial}{\partial\theta}f_{2k}\left(re^{i\theta}\right)}+(1-\lambda)}\right\}>\alpha \qquad (z\in U) \qquad (1.5)$$

where $k\geq 2$ is a fixed positive integer and $z=re^{i\theta},\ 0\leq\theta\leq 2\pi,\ 0\leq r<1,\ 0\leq\alpha<1$ and f_{2k} given by (1.4).

further, for functions f given by (1.1) and $g(z)=z+\sum_{n=2}^{\infty}A_nz^n+\overline{\sum_{n=1}^{\infty}B_nz^n}$, let $(f*g)(z)$ denote the Hadamard product (or convolution) of f ang g, defined by

$$(f*g)(z)=f(z)*g(z)=z+\sum_{n=2}^{\infty}A_na_nz^n+\overline{\sum_{n=1}^{\infty}B_nb_nz^n}.$$

In this paper, we determine a convolution characterization for functions in $P_{H_{sc}}^{(k)}(\lambda,\alpha)$. We then introduce a sufficient condition for functions to be in $P_{H_{sc}}^{(k)}(\lambda,\alpha)$ and $Q_{H_{sc}}^{(k)}(\lambda,\alpha)$.

Here we state a results due to [2], which will be used throughout this paper.

Lemma 1.1 Let $f=h+\overline{g}$ with h and g are given by (1.1). If

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n|+\frac{n+\alpha}{1-\alpha}|b_n|\right)\leq 2,$$

for $a_1 = 1,\ 0 \le \alpha < 1$. then f is sense-preserving, harmonic univalent in U, and $f \in \mathrm{P}_H(0,\alpha)$.

Lemma 1.2 Let $f = h + \overline{g}$ with h and g are given by (1.1). If

$$\sum_{n=1}^{\infty}\left(\frac{n(n-\alpha)}{1-\alpha}|a_n| + \frac{n(n+\alpha)}{1-\alpha}|b_n|\right) \le 2$$

,for $a_1 = 1,\ 0 \le \alpha < 1$. Then f is sense-preserving, harmonic univalent in U, and $f \in Q_H(0,\alpha)$.

2 Convolution Characterization

First , we give a meaningful conclusion about the class $\mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$

Theorem 2.1 A harmonic function $f = h + \overline{g}$ is in $\mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$ if and only if

$$\frac{1}{z}\left\{h*\left[\frac{z}{(1-z)^2}\left(\left[1-e^{i\theta}\right]-\lambda\left[1+(1-2\alpha)e^{i\theta}\right]\right)-\frac{(1-\lambda)\left[1+(1-2\alpha)e^{i\theta}\right]}{2}\varphi\right](z)\right.$$

$$-\frac{(1-\lambda)\left[1+(1-2\alpha)e^{i\theta}\right]}{2}\overline{(h*\varphi)(\overline{z})}$$

$$-g*\left[\frac{z}{(1-z)^2}\left(\left[1-e^{i\theta}\right]+\lambda\left[1+(1-2\alpha)e^{i\theta}\right]\right)+\frac{(1-\lambda)\left[1+(1-2\alpha)e^{i\theta}\right]}{2}\varphi\right](z)$$

$$\left.-\frac{(1-\lambda)\left[1+(1-2\alpha)e^{i\theta}\right]}{2}\overline{(g*\varphi)(\overline{z})}\right\} \ne 0, \qquad (2.1)$$

for all $z \in U$, $0 \le \alpha < 1$ and $0 \le \theta \le 2\pi$ where φ is given by (2.6).

Proof: For $0 \le \alpha < 1$, a harmonic function $f = h + \overline{g}$ is in $\mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$ if and only if the condition (1.3) holds. Differentiating $f\left(re^{i\theta}\right)$ with respect to θ and substituting in (1.3) we obtain

$$\Re\left\{\frac{\dfrac{zh'(z)-\overline{zg'(z)}}{f_{2k}(z)}}{\lambda\left[\dfrac{zh'(z)-\overline{zg'(z)}}{f_{2k}(z)}\right]+(1-\lambda)}\right\} > \alpha$$

.Or equivalent,

$$\frac{\dfrac{zh'(z)-\overline{zg'(z)}}{f_{2k}(z)}}{\lambda\left[\dfrac{zh'(z)-\overline{zg'(z)}}{f_{2k}(z)}\right]+(1-\lambda)} \neq \frac{1+(1-2\alpha)e^{i\theta}}{1-e^{i\theta}} \qquad (2.2)$$

For all $z \in U$, $0 \le \alpha < 1$ and $0 \le \theta \le 2\pi$ and the condition (2.2) can be written as

$$\frac{1}{z}\left\{\left(1-e^{i\theta}\right)\left[zh'(z)-\overline{zg'(z)}\right]-\left\{\lambda\left[zh'(z)-\overline{zg'(z)}\right]+\right.\right.$$

$$\left.\left.(1-\lambda)\left[h_{2k}(z)+\overline{g_{2k}(z)}\right]\right\}\left[1+(1-2\alpha)e^{i\theta}\right]\right\} \neq 0 \qquad (2.3)$$

On the other hand, it is well known that

$$zh'(z)=h(z)*\frac{z}{(1-z)^2}, \quad zg'(z)=g(z)*\frac{z}{(1-z)^2}. \qquad (2.4)$$

And from the definition of h_{2k} and g_{2k}, we know

$$h_{2k}=\frac{1}{2}\left[(h*\varphi)(z)+\overline{(h*\varphi)(\bar{z})}\right]; \quad g_{2k}=\frac{1}{2}\left[(g*\varphi)(z)+\overline{(g*\varphi)(\bar{z})}\right] \qquad (2.5)$$

Where $\varphi=\frac{1}{k}\sum_{v=0}^{k-1}\frac{z}{1-\varepsilon^{v}z}$. (2.6)

Substituting (2.4) and (2.5) in (2.3), we can get (2.1) easily. This completes the proof of the Theorem.

3 Coefficient Inequality

In this section, we give a sufficient condition for harmonic functions in $\mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$.

Theorem 3.1 Let $f=h+\bar{g}$ with h and g given by (1.1) and $f_{2k}=h_{2k}+\overline{g_{2k}}$ given by

(1.4). Let $\sum_{n=1}^{\infty}\left[\frac{\left|\left[(n-1)k+1\right](1-\lambda\alpha)a_{(n-1)k+1}-(1-\lambda)\Re\left(a_{(n-1)k+1}\right)\right|}{1-\alpha}\right.$

$$+\frac{\left[1-\alpha(1-\lambda)\right]\left|\Re\left(a_{(n-1)k+1}\right)\right|}{1-\alpha}-\frac{\left|\left[(n-1)k+1\right](1-\lambda\alpha)b_{(n-1)k+1}-(1-\lambda)\Re\left(b_{(n-1)k+1}\right)\right|}{1-\alpha}$$

$$+\frac{\left[1+\alpha(1-\lambda)\right]\left|\Re\left(b_{(n-1)k+1}\right)\right|}{1-\alpha}-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n(1-\alpha\lambda)}{1-\alpha}\left[|a_n|+|b_n|\right]\leq 2 \qquad (3.1)$$

where $a_1=1,\ 0\leq\alpha<1,\ l\geq 1$ and $k\geq 2$. Then f is sense-preserving, harmonic univalent in U and $f\in \mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$.

Proof: Since $\sum_{n=1}^{\infty}\left[\frac{n-\alpha}{1-\alpha}|a_n|+\frac{n+\alpha}{1+\alpha}|b_n|\right]$

$$\leq\sum_{n=1}^{\infty}\left\{\frac{\left|n(1-\alpha\lambda)a_n-(1-\lambda)\Re(a_n)\Phi_n\right|+\left[1-\alpha(1-\lambda)\right]\left|\Re(a_n)\right|\Phi_n}{1-\alpha}\right.$$

$$\left.+\frac{\left|n(1-\alpha\lambda)b_n-(1-\lambda)\Re(b_n)\Phi_n\right|+\left[1+\alpha(1-\lambda)\right]\left|\Re(b_n)\right|\Phi_n}{1-\alpha}\right\}$$

$$\left(\Phi_n=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-1)v},\quad \varepsilon=\exp(2\pi i/k)\right)$$

$$=\sum_{n=1}^{\infty}\left\{\frac{\left|(nk+1)(1-\alpha\lambda)a_{nk+1}-(1-\lambda)\Re(a_{nk+1})\right|+\left[1-\alpha(1-\lambda)\right]\left|\Re(a_{nk+1})\right|}{1-\alpha}\right.$$

$$\left.+\frac{\left|(nk+1)(1-\alpha\lambda)b_{nk+1}-(1-\lambda)\Re(b_{nk+1})\right|+\left[1+\alpha(1-\lambda)\right]\left|\Re(b_{nk+1})\right|}{1-\alpha}\right\}$$

$$-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n(1-\alpha\lambda)}{1-\alpha}\left[|a_n|+|b_n|\right]\leq 2,$$

by Lemma 1.1, we conclude that f is sense-preserving harmonic univalent and starlike in U. To proof $f\in \mathrm{P}_{H_{sc}}^{(k)}(\lambda,\alpha)$ according to the condition (1.3), we need to show that

$$\Re\left\{\frac{\dfrac{\dfrac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f_{2k}\left(re^{i\theta}\right)}}{\lambda\dfrac{\dfrac{\partial}{\partial\theta}f\left(re^{i\theta}\right)}{f_{2k}\left(re^{i\theta}\right)}+(1-\lambda)}\right\}=\Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{\lambda\left[zh'(z)-\overline{zg'(z)}\right]+(1-\lambda)\left[h_{2k}(z)+\overline{g_{2k}(z)}\right]}\right\}$$

$$=\Re\frac{A(z)}{B(z)}>\alpha,$$

where $z=re^{i\theta},\ 0\le\theta\le 2\pi,\ 0\le r<1,\ 0\le\alpha<1,\ 0\le\lambda<1$ and $k\ge 2$.

$$A(z)=zh'(z)-\overline{zg'(z)}=z+\sum_{n=2}^{\infty}na_nz^n-\overline{\sum_{n=1}^{\infty}nb_nz^n} \quad (3.2)$$

and

$$B(z)=\lambda\left[zh'(z)-\overline{zg'(z)}\right]+(1-\lambda)\left[h_{2k}(z)+\overline{g_{2k}(z)}\right]$$

$$=\lambda\left\{z+\sum_{n=2}^{\infty}na_nz^n-\overline{\sum_{n=1}^{\infty}nb_nz^n}\right\}+(1-\lambda)\left\{z+\sum_{n=2}^{\infty}\Re(a_n)\Phi_nz^n-\overline{\sum_{n=1}^{\infty}\Re(b_n)\Phi_nz^n}\right\} \quad (3.3)$$

where $\Phi_n=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-1)v},\ \ \varepsilon^k=1$

Using the fact that $\Re(w)\ge\alpha$ if and only if $|1-\alpha+w|\ge|1+\alpha-w|$, it suffices to show that $|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\ge 0$

On the other hand, for $A(z)$ and $B(z)$ as given in (3.2) and (3.3) respectively, we have $|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$

$$=\left|(1-\lambda)(1-\alpha)h_{2k}(z)+\left[1+(1-\alpha)\lambda\right]zh'(z)+\overline{(1-\lambda)(1-\alpha)g_{2k}(z)}\right.$$

$$\left.\overline{-\left[1+(1-\alpha)\lambda\right]zg'(z)}\right|$$

$$-\left|-(1-\lambda)(1+\alpha)h_{2k}(z)+\left[1-(1+\alpha)\lambda\right]zh'(z)-\overline{(1-\lambda)(1+\alpha)g_{2k}(z)}\right.$$

$$\overline{-\left[1-(1+\alpha)\lambda\right]zg'(z)}\Big| = \Big|(2-\alpha)z + \sum_{n=2}^{\infty}\left\{na_n\left[1-(1-\alpha)\lambda\right]+(1-\lambda)(1-\alpha)\Re(a_n)\Phi_n\right\}z^n$$

$$\overline{-\sum_{n=1}^{\infty}\left\{nb_n\left[1-(1-\alpha)\lambda\right]-(1-\lambda)(1-\alpha)\Re(b_n)\Phi_n\right\}z^n}\Big|$$

$$-\Big|-\alpha z + \sum_{n=2}^{\infty}\left\{na_n\left[1-(1+\alpha)\lambda\right]+(1-\lambda)(1+\alpha)\Re(a_n)\Phi_n\right\}z^n$$

$$\overline{-\sum_{n=1}^{\infty}\left\{nb_n\left[1-(1+\alpha)\lambda\right]+(1-\lambda)(1+\alpha)\Re(b_n)\Phi_n\right\}z^n}\Big|$$

$$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}\left|na_n\left[1+(1-\alpha)\lambda\right]+(1-\lambda)(1-\alpha)\Re(a_n)\Phi_n\right||z^n|$$

$$-\sum_{n=1}^{\infty}\left|nb_n\left[1+(1-\alpha)\lambda\right]-(1-\lambda)(1-\alpha)\Re(b_n)\Phi_n\right||z^n|$$

$$-\alpha|z| - \sum_{n=2}^{\infty}\left|na_n\left[1-(1+\alpha)\lambda\right]+(1-\lambda)(1+\alpha)\Re(a_n)\Phi_n\right||z^n|$$

$$-\sum_{n=1}^{\infty}\left|nb_n\left[1-(1+\alpha)\lambda\right]+(1-\lambda)(1+\alpha)\Re(b_n)\Phi_n\right||z^n|$$

$$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}\Big[\left|na_n\left[1+(1-\alpha)\lambda\right]+(1-\lambda)(1-\alpha)\Re(a_n)\Phi_n\right|$$

$$+\left(2-\alpha(1-\lambda)\right)|\Re(a_n)|\Phi_n\Big]|z^n|$$

$$-\sum_{n=1}^{\infty}\Big[\left|nb_n\left[1+(1-\alpha)\lambda\right]-(1-\lambda)\Re(b_n)\Phi_n\right|+(1-\lambda)\alpha|\Re(b_n)|\Phi_n\Big]|z^n|$$

$$-\alpha|z| - \sum_{n=2}^{\infty}\Big[\left|na_n\left[1-(1+\alpha)\lambda\right]-(1-\lambda)\Re(a_n)\Phi_n\right|-(1-\lambda)\alpha|\Re(a_n)|\Phi_n\Big]|z^n|$$

$$-\sum_{n=1}^{\infty}\Big[\left|nb_n\left[1-(1+\alpha)\lambda\right]-(1-\lambda)\Re(b_n)\Phi_n\right|+\left(2+(1-\lambda)\alpha\right)|\Re(b_n)|\Phi_n\Big]|z^n|$$

$$= 2(1-\alpha)|z|\{1$$

$$-\sum_{n=2}^{\infty}\frac{\left|n(1-\alpha\lambda)a_n-(1-\lambda)\Re(a_n)\Phi_n\right|+\left[1-\alpha(1-\lambda)\right]\left|\Re(a_n)\right|\Phi_n}{1-\alpha}\left|z^{n-1}\right|$$

$$\left.-\sum_{n=1}^{\infty}\frac{\left|n(1-\alpha\lambda)b_n-(1-\lambda)\Re(b_n)\Phi_n\right|+\left[1+\alpha(1-\lambda)\right]\left|\Re(b_n)\right|\Phi_n}{1-\alpha}\left|z^{n-1}\right|\right\}$$

$$\geq 2(1-\alpha)|z|\{1$$

$$-\sum_{n=2}^{\infty}\frac{\left|n(1-\alpha\lambda)a_n-(1-\lambda)\Re(a_n)\Phi_n\right|+\left[1-\alpha(1-\lambda)\right]\left|\Re(a_n)\right|\Phi_n}{1-\alpha}$$

$$\left.-\sum_{n=1}^{\infty}\frac{\left|n(1-\alpha\lambda)b_n-(1-\lambda)\Re(b_n)\Phi_n\right|+\left[1+\alpha(1-\lambda)\right]\left|\Re(b_n)\right|\Phi_n}{1-\alpha}\right\}$$

From the definition of Φ_n we know $\Phi_n=\begin{cases}1, & n=lk+1,\\ 0, & n\neq lk+1\end{cases}$ (3.4)

Substituting (3.4) in the last inequality above, we have

$$\left|A(z)+(1-\alpha)B(z)\right|-\left|A(z)-(1+\alpha)B(z)\right|$$

$$\geq 2(1-\alpha)|z|\{1$$

$$-\sum_{n=2}^{\infty}\frac{\left|(nk+1)(1-\alpha\lambda)a_{nk+1}-(1-\lambda)\Re(a_{nk+1})\right|+\left[1-\alpha(1-\lambda)\right]\left|\Re(a_{nk+1})\right|}{1-\alpha}$$

$$-\sum_{n=1}^{\infty}\frac{\left|(nk+1)(1-\alpha\lambda)b_{nk+1}-(1-\lambda)\Re(b_{nk+1})\right|+\left[1+\alpha(1-\lambda)\right]\left|\Re(b_{nk+1})\right|}{1-\alpha}$$

$$\left.-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n(1-\alpha\lambda)}{1-\alpha}|a_n|-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n(1-\alpha\lambda)}{1-\alpha}|b_n|-\frac{\left[1+\alpha(1-\lambda)\right]}{1-\alpha}|b_1|\right\}$$

$$= 2(1-\alpha)|z|\{2$$

$$-\sum_{n=1}^{\infty}\left[\frac{\left|\left[(n-1)k+1\right](1-\alpha\lambda)a_{(n-1)k+1}-(1-\lambda)\Re\left(a_{(n-1)k+1}\right)\right|}{1-\alpha}\right.$$

$$+\frac{\left[1-\alpha(1-\lambda)\right]\left|\Re\left(a_{(n-1)k+1}\right)\right|}{1-\alpha}$$

$$-\frac{\left|\left[(n-1)k+1\right](1-\alpha\lambda)b_{(n-1)k+1}-(1-\lambda)\Re\left(b_{(n-1)k+1}\right)\right|}{1-\alpha}$$

$$\left.+\frac{\left[1-\alpha(1-\lambda)\right]\left|\Re\left(b_{(n-1)k+1}\right)\right|}{1-\alpha}\right]$$

$$\left.-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n(1-\alpha\lambda)}{1-\alpha}\left[|a_n|+|b_n|\right]\right\}\geq 0,$$ by (3.1).

The Class $Q_{H_{sc}}^{(k)}(\lambda,\alpha)$

The technique of prove the following theorem for the class $Q_{H_{sc}}^{(k)}(\lambda,\alpha)$, is the same as the proof of Theorem 3.1 and by using Lemma 1.2 instead of Lemma 1.1.

Theorem 4.1 Let $f=h+\overline{g}$ with h and g given by (1.1) and $f_{2k}=h_{2k}+\overline{g_{2k}}$ given by (1.4). Let

$$\sum_{n=1}^{\infty}\left[\frac{\left[(n-1)k+1\right]\left[\left|\left[(n-1)k+1\right](1-\alpha\lambda)a_{(n-1)k+1}-(1-\lambda)\Re\left(a_{(n-1)k+1}\right)\right|\right]}{1-\alpha}\right.$$

$$+\frac{\left[1-\alpha(1-\lambda)\right]\left|\Re\left(a_{(n-1)k+1}\right)\right|}{1-\alpha}$$

$$-\frac{\left[(n-1)k+1\right]\left[\left|\left[(n-1)k+1\right](1-\alpha\lambda)b_{(n-1)k+1}-(1-\lambda)\Re\left(b_{(n-1)k+1}\right)\right|\right]}{1-\alpha}$$

$$+\frac{\left[1+\alpha(1-\lambda)\right]\left|\Re\left(b_{(n-1)k+1}\right)\right|}{1-\alpha}\Bigg]\Bigg\}-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n(1-\alpha\lambda)}{1-\alpha}\left[|a_n|+|b_n|\right]\Bigg\}\leq 2,$$

where $a_1=1,\ 0\leq\alpha<1,\ l\geq 1$ and $k\geq 2$. Then f is sense-preserving, harmonic univalent in U and $f\in Q_{H_{sc}}^{(k)}(\lambda,\alpha)$.

فئة جديدة للدوال الأحادية التوافقية المتعلقة ب $2k$ من النقاط المترافقة المتماثلة

سالمة فرج رمضان ناجي ماسلينا داروس

الملخص: لنأخذ S_H تعرف فئة من الدوال $f=h+\bar{g}$ الأحادية التوافقية والمحافظة في قرص الوحدة U. في هذه الورقة نحن نقدم فئات جديدة من الدوال الأحادية التوافقية المتعلقة ب $2k$ من النقاط المترافقة المتماثلة. العديد من الخواص علي هذه الفئات تم دراستها في هذه الورقة.

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